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Interpreting Observables in a Quantum World from the Categorial Standpoint

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We develop a relativistic perspective on structures of quantum observables, in terms of localization systems of Boolean coordinatizing charts. This perspective implies that the quantum world is comprehended via Boolean reference frames for measurement of observables, pasted together along their overlaps. The scheme is formalized categorically, as an instance of the adjunction concept. The latter is used as a framework for the specification of a categorical equivalence signifying an invariance in the translational code of communication between Boolean localizing contexts and quantum systems. Aspects of the scheme semantics are discussed in relation to logic. The interpretation of coordinatizing localization systems, as structure sheaves, provides the basis for the development of an algebraic differential geometric machinery suited to the quantum regime.

KEY WORDS: Boolean; quantum; observables; localization; adjunction; topos; sheaves; quantum logic; abstract differential geometry.

1. INTRODUCTION

In the working understanding of physical theories the concept of observables is associated with physical quantities, that in principle, can be measured. Quantum theory stipulates that quantities admissible as measured results must be real numbers. The resort to real numbers has the advantage of making our empirical access secure, since real number representability consists our form of observation. In any experiment performed by an observer, the propositions that can be made concerning a physical quantity are of the type, which asserts that, the value of the physical quantity lies in some Borel set of the real numbers. The proposition that the value of a physical quantity lies in a Borel set of the real line corresponds to an event in the ordered event structure of the theory, as it is apprehended by an observer. Thus we obtain a mapping from the Borel sets of the real line to the event structure which captures precisely the notion of observable:

$$Z : \mathrm{Bor}(\mathbf{R}) \to L$$

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Most importantly the above mapping is required to be a homomorphism. In this representation Bor($\mathbf{R}$) stands for the algebra of events associated with a measurement device interacting with a physical system. The homomorphism assigns to every empirical event in Bor($\mathbf{R}$) a proposition or event in L, that states, a measurement fact about the physical system interacting with the measuring apparatus. We may argue that the real line endowed with its Borel structure serves as a modeling object, which schematizes the event algebra of an observed system, by projecting into it its structure. In the Hilbert space formalism of quantum theory, events are considered as closed subspaces of a seperable, complex Hilbert space corresponding to a physical system. Then the quantum event algebra is identified with the lattice of closed subspaces of the Hilbert space, ordered by inclusion and carrying an orthocomplementation operation which is given by the orthogonal complements of the closed subspaces (Birkhoff and von Neumann, 1936; Varadarajan, 1968).

In this work we will develop the idea that in quantum theory, Boolean observables can be understood as providing a coordinatization of the quantum world by establishing a relativity principle. An intuitive flavor of this insight is provided by Kochen–Specker theorem (Kochen and Specker, 1967), according to which the complete comprehension of a quantum mechanical system is impossible, in case that, a single system of Boolean devices is only used. On the other side, in every concrete measurement context, the set of events that have been actualized in this context forms a Boolean algebra. Hence it is reasonable to assert that a Boolean observable picks a specific Boolean algebra, which may be considered, as a Boolean subalgebra of the quantum lattice of events. Essentially, we may argue that, a Boolean observable schematizes the quantum event structure by correlating its Boolean subalgebras picked by measurements with the smallest Boolean algebra containing all the clopen sets of the real line. According to the assertion above, Boolean observables play the role of coordinatizing objects in the attempt to probe the quantum world. This is equivalent to the statement that a Boolean algebra in the lattice of quantum events picked by an observable, serves as a reference frame, conceived in a precise category-theoretical sense, relative to which the measurement result is being coordinatized, pointing at the same time, towards a contextualistic perspective on the structure of quantum events. The philosophical meaning of the proposed scheme implies that the quantum world is being perceived through Boolean reference frames, regulated by our measurement procedures, which interlock to form a coherent picture in a nontrivial way.

In this work we propose a mathematical scheme for the implementation of the above thesis based on category-theoretical methods (Bell, 1988; Lawvere and Schanuel, 1997; MacLane, 1971; MacLane and Moerdijk, 1992). The main guiding idea in our investigation consists of the use of objects belonging to the Boolean species of observable structure, as modeling figures, for probing the objects belonging to the quantum species of observable structure. The category-theoretical interpretational framework provides the appropriate means to implement this idea in a universal way. The Boolean event algebras modeling objects, being formed by observational procedures, give rise to Boolean localization systems, which, in turn, provide structure preserving maps from the domain of variable Boolean probing objects to quantum algebras of events. Subsequently, under suitable compatibility relations, it is possible to obtain an isomorphism between quantum event algebras and Boolean localization systems for observables. The essence of this scheme is the development of a Boolean manifold perspective on quantum event structures, according to which, a quantum event algebra consists an interconnected family of Boolean ones intelocking in a nontrivial fashion. The physical interpretation of the Boolean manifold scheme takes place through the identification of Boolean charts in systems of measurement localization for quantum event algebras with reference frames of a topos-theoretical nature, relative to which, the results of measurements can be coordinatized. Thus any Boolean chart in a covering atlas for a quantum algebra of events corresponds to a set of Boolean events which become realizable in the experimental context of it. The above identification is equivalent to the introduction of a relativity principle in quantum theory and suggests a contextualistic interpretation of its formalism.

Summing up, the main thesis of this paper, is that, the quantum world is being perceived through Boolean reference frames, objectified by measuring arrangements being set up experimentally, that can be pasted together using category-theoretical means. Contextual topos-theoretical approaches to quantum structures have been also considered, from a differnt viewpoint in literatures (Butterfield and Isham, 1998, 1999), and discussed in literatures (Butterfield and Isham, 2000; Raptis, 2001; Rawling and Selesnick, 2000).

In Section 2, we introduce the categories associated with observable structures. In Section 3, we construct Boolean shaping and Boolean presheaf observable functors, and also develop the idea of fibrations over Boolean observables. In Section 4, we prove the existence of an adjunction between the topos of presheaves of Boolean observables and the category of quantum observables. In Section 5, we analyze this adjoint situation and show that the adjunctive correspondence is based on a tensor product construction. In Section 6, the notion of systems of localization for measurement of observables over a quantum event algebra is being introduced and analyzed. In Section 7, we establish the representation of quantum event algebras as manifolds of Boolean measurement localization systems. In Section 8 we develop the semantics associated with Boolean localization systems. In Section 9, we comment on the logical implications of the Boolean manifold scheme with respect to the category-theoretical framework. Finally, in Section 10, we develop some ideas related with the exploitation of the proposed model in the direction of developing an appropriate differential geometric machinery in the quantum regime, based on sheaf theoretical methods. We, finally, conclude in Section 11.

2. CATEGORIES ASSOCIATED WITH OBSERVABLES

According to the category-theoretical approach to each species of mathematical structure, there corresponds a *category* whose objects have that structure, and whose morphisms preserve it. Moreover to any natural construction on structures of one species, yielding structures of another species, there corresponds a *functor* from the category of first species to the category of the second.

A *classical event structure* is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. Its objects are Boolean algebras of events, and its arrows are Boolean algebraic homomorphisms.

A *quantum event structure* is a small category, denoted by $\mathcal{L}$, which is called the category of quantum event algebras.

Its objects are quantum algebras of events, that is, partially-ordered sets of quantum events, endowed with a maximal element 1, and with an operation of orthocomplementation $[-]^*$: $L \to L$, which satisfy, for all $l \in L$ the following conditions: (a) $l \leq 1$, (b) $l^{**} = l$, (c) $l \vee l^* = 1$, (d) $l \leq l' \Rightarrow l'^* \leq l^*$, (e) $l \perp l' \Rightarrow l \vee l' \in L$, (f) $l \vee l' = 1$, $l \wedge l' = 0 \Rightarrow l = l'^*$, where $0 := 1^*$, $l \perp l' := l \leq l'^*$, and the operations of meet $\wedge$ and join $\vee$ are defined as usually.

Its arrows are quantum algebraic homomorphisms, that is maps $L \xrightarrow{H} K$, which satisfy, for all $k \in K$ the following conditions: (a) $H(1) = 1$, (b) $H(k^*) = [H(k)]^*$, (c) $k \leq k' \Rightarrow H(k) \leq H(k')$, (d) $k \perp k' \Rightarrow H(k \vee k') \leq H(k) \vee H(k')$.

Next we introduce the categories associated with structure of observables.

A *quantum observable space structure* is a small category, denoted by $\mathcal{O}B$, which is called the category of spaces of quantum observables.

Its objects are the sets Ω of real-valued observables on a quantum event algebra L, where each observable Ξ is defined to be an algebraic homomorphism from the Borel algebra of the real line Bor($\mathbf{R}$), to the quantum event algebra L

$$\Xi : \mathrm{Bor}(\mathbf{B}) \to L$$

such that the following conditions are satisfied:

(i) $\Xi(\emptyset) = 0$, $\Xi(\mathbf{R}) = 1$, (ii) $E \cap F = \emptyset \Rightarrow \Xi(E) \perp \Xi(F)$, for $E, F \in \mathrm{Bor}(\mathbf{R})$, (iii) $\Xi(\cup_n E_n) = \vee_n \Xi(E_n)$, where $E_1, E_2, \ldots$ sequence of mutually disjoint Borel sets of the real line.

If L is isomorphic with the orthocomplemented lattice of orthogonal projections on a Hilbert space, then it follows from von Neumann's spectral theorem that the observables are in 1-1 correspondence with the hypermaxmimal Hermitian operators on the Hilbert space.

Moreover each set Ω is endowed with a right action $R : \Omega \times \mathrm{Bor}\, f(\mathbf{R}) \to \Omega$ from the semigroup of all real-valued Borel functions of a real variable $f : \mathbf{R} \to \mathbf{R}$ which satisfy the following condition:

$$E \in \mathrm{Bor}(R) \Rightarrow f^{-1}(E) \in \mathrm{Bor}(\mathbf{R})$$

According to the above we have

$$(\Xi, f) \in \Omega \times \operatorname{Bor} f(\mathbf{R}) \mapsto \Xi \bullet f = \Xi(f^{-1}(E)) \in \Omega$$

To sum up the objects of the category of quantum observables are the spaces $\boldsymbol{\Omega} = < \Omega, \mathbf{R} >$ of real-valued observables.

Its arrows are the quantum observable spaces homomorphisms $h : \boldsymbol{\Omega} \rightarrow \mathbf{U}$, namely set-homomorphisms $[]^h : \Omega \rightarrow U$ which respect the right action of $\operatorname{Bor} f(\mathbf{R})$:

$$[\Xi \bullet f]^h = \Xi^h \bullet f$$

We note that $\boldsymbol{\Omega}$ and $\mathbf{U}$ are regarded as defined over the same quantum event algebra L, otherwise we have to take into account the quantum algebraic homomorphisms as well.

Using the information encoded in the categories of quantum event algebras $\mathcal{L}$, and spaces of quantum observables $\mathcal{O}B$, it is possible to construct a new category, called the category of quantum observables, which is going to play a key role in the subsequent analysis.

For this purpose it is appropriate to introduce, first of all, the notion of the slice category $[Q/\mathcal{L}]$, where Q is an object in the category of quantum event algebras $\mathcal{L}$. The slice category $[Q/\mathcal{L}]$ is characterized as the category whose objects are quantum algebraic homomorphisms $E : Q \rightarrow L$ and whose arrows $E \longrightarrow Z$ are the commutative triangles (Diagram 1).

We observe that there exists an obvious forgetful functor

$$\mathbf{F} : [Q/\mathcal{L}] \rightarrow \mathcal{L}$$

such that: $\mathbf{F}(E : Q \rightarrow L) = L$. The forgetful functor $\mathbf{F}$, makes $[Q/\mathcal{L}]$ into a fibered category over $\mathcal{L}$. By the latter, we mean, in general the following.

A category fibered over $\mathcal{L}$, is a category $\mathcal{V}$ and a functor $\mathbf{F} : V \rightarrow \mathcal{L}$ such that:

(i) If we are given any quantum algebraic homomorphism $\Gamma : L' \rightarrow L$ in $\mathcal{L}$ and an object V of $\mathcal{V}$, with $\mathbf{F}(V) = L$, there exists a quantum algebraic homomorphism $\Delta : V' \rightarrow V$ such that $\mathbf{F}(\Delta) = \Gamma$.

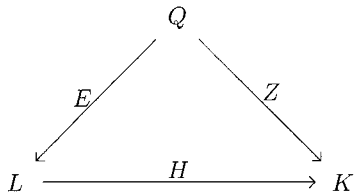

Diagram 1

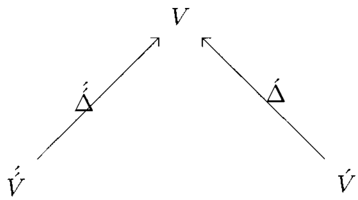

Diagram 2

(ii) If we are given (Diagram 2) and (Diagram 3), in $\mathcal{V}$ and $\mathcal{L}$ correspondingly, then there exists a unique arrow $\Sigma : V'' \to V'$ such that $\Delta' \circ \Sigma = \Delta''$, and $\mathbf{F}(\Sigma) = \Phi$.

We notice, that given a quantum event algebra L in $\mathcal{L}$, the fiber $\mathbf{F}_L$ is characterized as the subcategory of $\mathcal{V}$, whose objects map to L, and whose arrows map to 1_L under the functor $\mathbf{F}$. Remarkably, each fiber $\mathbf{F}_L$ is a groupoid, or else, every morphism in $\mathbf{F}_L$ is an invertible one.

Now, if we consider the forgetful functor $\mathbf{F} : [Q/\mathcal{L}] \to \mathcal{L}$, it is straightforward to show that $[Q/\mathcal{L}]$ is a category fibered over $\mathcal{L}$ in groupoids.

For, given a quantum algebraic homomorphism $\Lambda : L' \to L$, and an object $E : Q \to L'$ of the slice category $[Q/\mathcal{L}]$, the composition $\Lambda \circ E : Q \to L$, is an object of $[Q/\mathcal{L}]$ with, $\mathbf{F}(\Lambda \circ E) = L$, and moreover, the quantum algebraic homomorphism Λ, is an arrow from E to $\Lambda \circ E$. In fact, this arrow is clearly the only arrow from E to $\Lambda \circ E$, whose image under $\mathbf{F}$ is Λ. The above makes conditions (i) and (ii) easily verifiable. Thus, the slice category $[Q/\mathcal{L}]$ is fibered in Sets over $\mathcal{L}$, or else, it constitutes a discretely fibered category. It is easy to see that, given any two objects of the slice category $[Q/\mathcal{L}]$, with the same codomain: $Z : Q \to L$ and $Z' : Q \to L$, such that $\mathbf{F}(Z) = \mathbf{F}(Z')$, or else, Z, Z' belong to the fiber $\mathbf{F}_L$, any arrow $\Upsilon : L' \to L$, such that $\mathbf{F}(\Upsilon) = 1_L$, must be the identity. We conclude that the fibers of the slice category $[Q/\mathcal{L}]$ are sets, or equivalently, may be considered as groupoids with the property that they have no nonidentity morphisms.

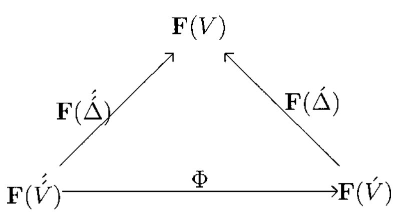

Diagram 3

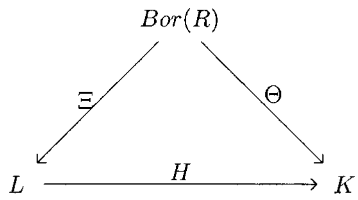

Diagram 4

Now, we may apply the previous discussion for the case $Q = \text{Bor}(\mathbf{R})$, since evidently, the Borel algebra of the real line can be considered as an object in $\mathcal{L}$, in order to obtain the slice category $[\text{Bor}\mathbf{R}/\mathcal{L}]$. The above is identified as the category of quantum observables, characterized as a category fibered in groupoids over the category of quantum event algebras $\mathcal{L}$. In more detail.

A *quantum observable structure* is a small category, denoted by $\mathcal{O}_Q$, which is called the category of quantum observables.

Its objects are the quantum observables $\Xi : \text{Bor}(\mathbf{R}) \rightarrow L$ and its arrows $\Xi \longrightarrow \Theta$ are the commutative triangles (Diagram 4), or equivalently the quantum algebraic homomorphisms $L \xrightarrow{H} K$ in $\mathcal{L}$, such that $\Theta = H \circ \Xi$ in (Diagram 4) is again a quantum observable.

Correspondingly, a *Boolean observable structure* is a small category, denoted by $\mathcal{O}_B$, which is called the category of Boolean observables.

Its objects are the Boolean observables $\xi : \text{Bor}(\mathbf{R}) \rightarrow B$ and its arrows are the Boolean algebraic homomorphisms $B \xrightarrow{h} C$ in $\mathcal{B}$, such that $\theta = h \circ \xi$ in Diagram 5 is again a Boolean observable.

We note parenthetically, that the categories $\mathcal{B}$, $\mathcal{L}$, $\mathcal{O}_B$, and $\mathcal{O}_Q$ are algebraic categories, and have arbitrary colimits (MacLane, 1971).

3. PRESHEAF AND COORDINATIZATION BOOLEAN OBSERVABLE FUNCTORS

3.1. Presheaves of Boolean Observables

If $\mathcal{O}_B^{op}$ is the opposite category of $\mathcal{O}_B$, then $\mathbf{Sets}^{\mathcal{O}_B^{op}}$ denotes the functor category of presheaves on Boolean observables. Its objects are all functors $\mathbf{X} : \mathcal{O}_B^{op} \longrightarrow$

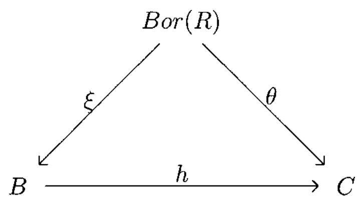

Diagram 5

Sets, and its morphisms are all natural transformations between such functors. Each object $\mathbf{X}$ in this category is a contravariant set-valued functor on $\mathcal{O}_B$, called a presheaf on $\mathcal{O}_B$.

A functor $\mathbf{X}$ is a structure-preserving morphism of these categories, that is it preserves composition and identities. A functor in the category $\mathbf{Sets}^{\mathcal{O}_B^{op}}$ can be understood as a contravariant translation of the language of $\mathcal{O}_B$ into that of **Sets**. Given another such translation (contravariant functor) $\mathbf{X}'$ of $\mathcal{O}_B$ into **Sets** we need to compare them. This can be done by giving, for each object ξ in $\mathcal{O}_B$ a transformation $\tau_\xi : \mathbf{X}(\xi) \longrightarrow \mathbf{X}'(\xi)$ which compares the two images of the object ξ. Not any morphism will do, however, as it would be necessary the construction to be parametric in ξ, rather than ad hoc. Since ξ is an object in $\mathcal{O}_B$ while $\mathbf{X}(\xi)$ is in **Sets** we cannot link them by a morphism. Rather the goal is that the transformation should respect the morphisms of $\mathcal{O}_B$, or in other words the interpretations of $v : \xi \longrightarrow \xi'$ by $\mathbf{X}$ and $\mathbf{X}'$ should be compatible with the transformation under τ. Then τ is a natural transformation in the presheaf category $\mathbf{Sets}^{\mathcal{O}_B^{op}}$.

For each Boolean observable ξ of $\mathcal{O}_B$, $\mathbf{X}(\xi)$ is a set, and for each arrow $f : \theta \longrightarrow \xi$, $\mathbf{X}(f) : \mathbf{X}(\xi) \longrightarrow \mathbf{X}(\theta)$ is a set function. If $\mathbf{X}$ is a presheaf on $\mathcal{O}_B$ and $x \in \mathbf{X}(\mathrm{O})$, the value $\mathbf{X}(f)(x)$ for an arrow $f : \theta \longrightarrow \xi$ in $\mathcal{O}_B$ is called the restriction of x along f and is denoted by $\mathbf{X}(f)(x) = x \circ f$.

Each object ξ of $\mathcal{O}_B$ gives rise to a contravariant Hom-functor $y[\xi] := \mathrm{Hom}_{\mathcal{O}_B}(-, \xi)$. This functor defines a presheaf on $\mathcal{O}_B$. Its action on an object θ of $\mathcal{O}_B$ is given by

$$y[\xi](\theta) := \mathrm{Hom}_{\mathcal{O}_B}(\theta, \xi)$$

whereas its action on a morphism $\eta \xrightarrow{w} \theta$, for $v : \theta \longrightarrow \xi$ is given by

$$y[\xi](w) : \mathrm{Hom}_{\mathcal{O}_B}(\theta, \xi) \longrightarrow \mathrm{Hom}_{\mathcal{O}_B}(\eta, \xi)$$

$$y[\xi](w)(v) = v \circ w$$

Furthermore y can be made into a functor from $\mathcal{O}_B$ to the contravariant functors on $\mathcal{O}_B$

$$y : \mathcal{O}_B \longrightarrow \mathbf{Sets}^{\mathcal{O}_B^{op}}$$

such that $\xi \mapsto \mathrm{Hom}_{\mathcal{O}_B}(-, \xi)$. This is an embedding and it is a full and faithful functor.

The functor category of presheaves on Boolean observables $\mathbf{Sets}^{\mathcal{O}_B^{op}}$, provides an instantiation of a structure known as topos. A topos exemplifies a well defined notion of variable set. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized space. Moreover it provides a natural example of a many-valued truth structure, which remarkably is not ad hoc, but reflects genuine constraints of the surrounding universe. The study of the truth value structure associated with the topos of presheaves of Boolean observables and its significance for a quantum logical interpretation of the proposed categorical scheme will be the subject of a separate paper. Some ideas related to this direction are discussed in Section 9.

3.2. The Category of Elements

Since $\mathcal{O}_B$ is a small category, there is a set consisting of all the elements of all the sets $\mathbf{X}(\xi)$, and similarly there is a set consisting of all the functions $\mathbf{X}(f)$. This observation regarding $\mathbf{X} : \mathcal{O}_B^{op} \longrightarrow$ **Sets** permits us to take the disjoint union of all the sets of the form $\mathbf{X}(\xi)$ for all objects ξ of $\mathcal{O}_B$. The elements of this disjoint union can be represented as pairs (ξ, x) for all objects ξ of $\mathcal{O}_B$ and elements $x \in \mathbf{X}(\xi)$. Thus the disjoint union of sets is made by labelling the elements. Now we can construct a category whose set of objects is the disjoint union just mentioned. This structure is called the category of elements of the presheaf $\mathbf{X}$, denoted by $\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B})$. Its objects are all pairs (ξ, x), and its morphisms $(\xi', x') \longrightarrow (\xi, x)$ are those morphisms $u : \xi' \longrightarrow \xi$ of $\mathcal{O}_B$ for which $xu = x'$. Projection on the second coordinate of $\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B})$, defines a functor $\mathbf{G}_\mathbf{X} : \mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B}) \longrightarrow \mathcal{O}_\mathbf{B}$. $\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B})$ together with the projection functor $\mathbf{G}_\mathbf{X}$ is equivalent to the discrete fibration induced by $\mathbf{X}$, and $\mathcal{O}_B$ is the base category of the fibration. We note that the fibration is discrete because the fibers are categories in which the only arrows are identity arrows (Diagram 6). If ξ is a Boolean observable object of $\mathcal{O}_B$, the inverse image under $\mathbf{G}_\mathbf{P}$ of ξ is simply the set $\mathbf{X}(\xi)$, although its elements are written as pairs so as to form a disjoint union. The instantiation of the fibration induced by $\mathbf{P}$, is an application of the general Grothendieck construction (Artin *et al.*, 1972).

3.3. Coordinatization Functor

We define a modelling or coordinatisation functor, $\mathbf{A} : \mathcal{O}_B \longrightarrow \mathcal{O}_Q$, which assigns to Boolean observables in $\mathcal{O}_B$, that instantiates a model category, the underlying quantum observables from $\mathcal{O}_Q$, and to Boolean homomorphisms the underlying quantum algebraic homomorphisms. Hence $\mathbf{A}$ acts as a forgetful functor, forgetting the extra Boolean structure of $\mathcal{O}_B$.

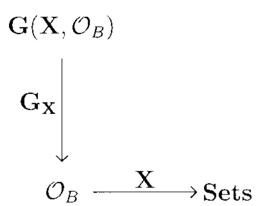

Diagram 6

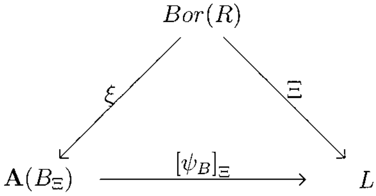

Diagram 7

Equivalently the coordinatization functor can be characterized as, $\mathbf{A} : \mathcal{B} \longrightarrow \mathcal{L}$, which assigns to Boolean event algebras in $\mathcal{B}$ the underlying quantum event algebras from $\mathcal{L}$, and to Boolean homomorphisms the underlying quantum algebraic homomorphisms, such that Diagram 7 commutes.

4. ADJOINTNESS BETWEEN PRESHEAVES OF BOOLEAN OBSERVABLES AND QUANTUM OBSERVABLES

We consider the category of quantum observables $\mathcal{O}_Q$ and the modelling functor $\mathbf{A}$, and we define the functor $\mathbf{R}$ from $\mathcal{O}_Q$ to the topos of presheaves given by

$$\mathbf{R}(\Xi) : \xi \mapsto \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

A natural transformation τ between the topos of presheaves on the category of Boolean observables $\mathbf{X}$ and $\mathbf{R}(\Xi)$, $\tau : \mathbf{X} \longrightarrow \mathbf{R}(\Xi)$ is a family τ_ξ indexed by Boolean observables ξ of $\mathcal{O}_B$ for which each τ_ξ is a map

$$\tau_\xi : \mathbf{X}(\xi) \to \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

of sets, such that the diagram of sets (Diagram 8) commutes for each Boolean homomorphism $u : \xi' \to \xi$ of $\mathcal{O}_B$.

If we make use of the category of elements of the Boolean observables-variable set X, being an object in the topos of presheaves, then the map τ_ξ, defined above, can be characterized as:

$$\tau_\xi : (\xi, p) \to \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A} \circ G_{\mathbf{X}}(\xi, p), \Xi)$$

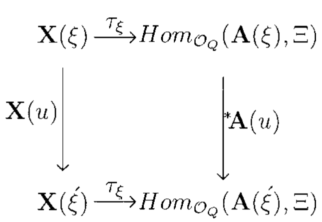

Diagram 8

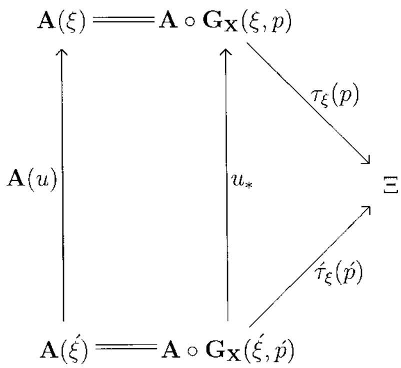

Diagram 9

Equivalently such a τ can be seen as a family of arrows of $\mathcal{O}_Q$ which is being indexed by objects (ξ, p) of the category of elements of the presheaf of Boolean observables **X**, namely

$$\{\tau_\xi(p) : \mathbf{A}(\xi) \to \Xi\}_{(\xi, p)}$$

From the perspective of the category of elements of **X**, the condition of the commutativity of Diagram 8 is equivalent with the condition that for each Boolean homomorphism $u : \xi' \to \xi$ of $\mathcal{O}_B$, Diagram 9 commutes.

From Diagram 9 we can see that the arrows $\tau_\xi(p)$ form a cocone from the functor $\mathbf{A} \circ G_\mathbf{X}$ to the quantum observable algebra object Ξ. Making use of the definition of the colimit, we conclude that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit **LX** to the quantum observable object Ξ. In other words, there is a bijection which is natural in **X** and Ξ.

$$\mathrm{Nat}(\mathbf{X}, \mathbf{R}(\Xi)) \cong \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{LX}, \Xi)$$

From the above bijection we are driven to the conclusion that the functor **R** from $\mathcal{O}_Q$ to the topos of presheaves given by

$$\mathbf{R}(\Xi) : \xi \mapsto \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

has a left adjoint $\mathbf{L} : \mathbf{Sets}^{\mathcal{O}_B^{op}} \to \mathcal{O}_Q$, which is defined for each presheaf of Boolean observables **X** in $\mathbf{Sets}^{\mathcal{O}_B^{op}}$ as the colimit

$$\mathbf{L}(\mathbf{X}) = \mathrm{Colim}\left\{\mathbf{G}(\mathbf{X}, \mathcal{O}_\mathbf{B}) \xrightarrow{\mathbf{G_X}} \mathcal{O}_\mathbf{B} \xrightarrow{\mathbf{A}} \mathcal{O}_\mathbf{Q}\right\}$$

Consequently there is a *pair of adjoint functors* $\mathbf{L} \dashv \mathbf{R}$ as follows:

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{O}_B^{op}} \leftrightarrows \mathcal{O}_Q : \mathbf{R}$$

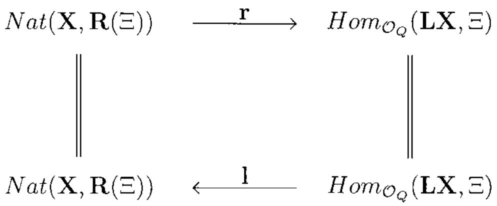

Diagram 10

Thus we have constructed an adjunction which consists of the functors **L** and **R**, called left and right adjoints with respect to each other respectively, as well as the natural bijection

$$\mathrm{Nat}(\mathbf{X}, \mathbf{R}(\Xi)) \cong \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{LX}, \Xi)$$

In the adjoint situation described above, between the topos of presheaves of Boolean observables and the category of quantum observables (Diagram 10), the map **r** is called the right adjunction operator and the map **1** the left adjunction operator.

If in the bijection defining the adjunction we use as **X** the representable presheaf of the topos of Boolean observables $\mathbf{y}[\xi]$, it takes the form:

$$\mathrm{Nat}(\mathbf{y}[\xi], \mathbf{R}(\Xi)) \cong \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{Ly}[\xi], \Xi)$$

We note that when $\mathbf{X} = \mathbf{y}[\xi]$ is representable, then the corresponding category of elements $\mathbf{G}(\mathbf{y}[\xi], \mathcal{O}_\mathbf{B})$ has a terminal object, namely the element $1 : \xi \longrightarrow \xi$ of $\mathbf{y}[\xi](\xi)$. Therefore the colimit of the composite $\mathbf{A} \circ \mathbf{G}_{\mathbf{y}[\xi]}$ is going to be just the value of $\mathbf{A} \circ \mathbf{G}_{\mathbf{y}[\xi]}$ on the terminal object. Thus we have

$$\mathbf{Ly}[\xi](\xi) \cong \mathbf{A} \circ \mathbf{G}_{\mathbf{y}[\xi]}(\xi, 1_\xi) = \mathbf{A}(\xi)$$

Thus we can characterize $\mathbf{A}(\xi)$ as the colimit of the representable presheaf on the category of Boolean observables (Diagram 11).

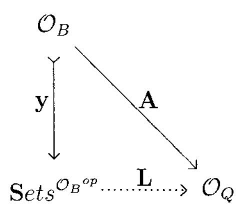

Diagram 11

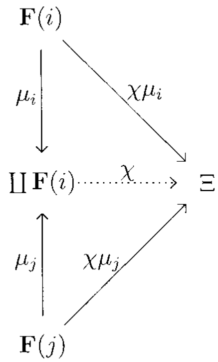

Diagram 12

5. ANALYSIS OF THE ADJUNCTION

The content of the adjunction between the topos of presheaves of Boolean observables and the category of quantum observables can be analyzed if we make use of the categorical construction of the colimit defined above, as a coequalizer of a coproduct. We consider the colimit of any functor $\mathbf{F} : I \longrightarrow \mathcal{O}_Q$ from some index category **I** to $\mathcal{O}_Q$. Let $\mu_i : \mathbf{F}(i) \to \coprod_i \mathbf{F}(i)$, $i \in I$, be the injections into the coproduct (Diagram 12). A morphism from this coproduct, $\chi : \coprod_i \mathbf{F}(i) \to \Xi$, is determined uniquely by the set of its components $\chi_i = \chi\mu_i$. These components χ_i are going to form a cocone over **F** to the quantum observable vertex Ξ only when for all arrows $v : i \longrightarrow j$ of the index category I the following conditions are satisfied

$$(\chi\mu_j)\mathbf{F}(v) = \chi\mu_i$$

So we consider all $\mathbf{F}(\mathrm{dom}v)$ for all arrows v with its injections ν_v and obtain their coproduct $\coprod_{v:i\to j} \mathbf{F}(\mathrm{dom}v)$. Next we construct two arrows ζ and η, defined in terms of the injections ν_v and μ_i, for each $v : i \longrightarrow j$ by the conditions

$$\zeta\nu_v = \mu_i$$

$$\eta\nu_v = \mu_j\mathbf{F}(v)$$

as well as their coequalizer χ (Diagram 13).

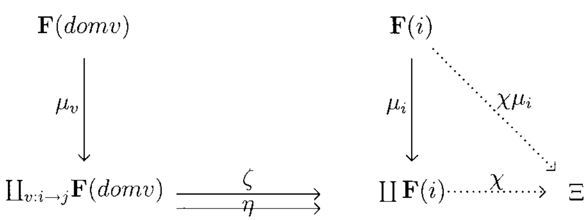

Diagram 13

$$\coprod_{v:i\to j}\mathbf{F}(domv) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod \mathbf{F}(i) \xrightarrow{\chi} Colim\mathbf{F}$$

Diagram 14

The coequalizer condition $\chi\zeta = \chi\eta$ tells us that the arrows $\chi\mu_i$ form a cocone over $\mathbf{F}$ to the quantum observable vertex $\mathcal{O}_Q$. We further note that since χ is the coequalizer of the arrows ζ and η this cocone is the colimiting cocone for the functor $\mathbf{F} : I \to \mathcal{O}_Q$ from some index category I to $\mathcal{O}_Q$. Hence the colimit of the functor $\mathbf{F}$ can be constructed as a coequalizer of coproduct according to Diagram 14.

In the case considered the index category is the category of elements of the presheaf of Boolean observables $\mathbf{X}$ and the functor $\mathbf{A} \circ G_{\mathbf{X}}$ plays the role of the functor $\mathbf{F} : I \longrightarrow \mathcal{O}_Q$. In the diagram above the second coproduct is over all the objects (ξ, p) with $p \in \mathbf{X}(\xi)$ of the category of elements, while the first coproduct is over all the maps $v : (\xi', p') \longrightarrow (\xi, p)$ of that category, so that $v : \xi' \longrightarrow \xi$ and the condition $pv = p'$ is satisfied. We conclude that the colimit $\mathbf{L}_A(P)$ can be equivalently presented as the coequalizer (Diagram 15).

The coequalizer presentation of the colimit shows that the Hom-functor $\mathbf{R}_A$ has a left adjoint which can be characterized categorically as the tensor product— $\otimes_{\mathcal{O}_B}\mathbf{A}$.

In order to clarify the above observation, we forget for the moment that the discussion concerns the category of quantum observables $\mathcal{O}_Q$, and we consider instead the category **Sets**. Then the coproduct, $\coprod_p \mathbf{A}(\xi)$ is a coproduct of sets, which is equivalent to the product $\mathbf{X}(\xi) \times \mathbf{A}(\xi)$ for $\xi \in \mathcal{O}_B$. The coequalizer is thus the definition of the tensor product $\mathcal{P} \otimes \mathcal{A}$ of the set valued factors:

$$\mathbf{X} : \mathcal{O}_B^{op} \longrightarrow \mathbf{Sets}, \qquad \mathbf{A} : \mathcal{O}_B \longrightarrow \mathbf{Sets}$$

According to Diagram 16, for elements $p \in \mathbf{X}(\xi)$, $v : \xi' \to \xi$ and $q' \in \mathbf{A}(\xi')$ the following equations hold:

$$\zeta(p, v, q') = (pv, q'), \qquad \eta(p, v, q') = (p, vq')$$

symmetric in $\mathbf{X}$ and $\mathbf{A}$. Hence the elements of the set $\mathbf{X} \otimes_{\mathcal{O}_B} \mathbf{A}$ are all of the form $\chi(p, q)$. This element can be written as

$$\chi(p, q) = p \otimes q, \qquad p \in \mathbf{X}(\xi), q \in \mathbf{A}(\xi)$$

Thus if we take into account the definitions of ζ and η above, we obtain.

$$\coprod_{v:\acute{\xi}\to\xi}\mathbf{A}(\acute{\xi}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{(\xi,p)}\mathbf{A}(\xi) \xrightarrow{\chi} \mathbf{X}\otimes_{\mathcal{O}_B}\mathbf{A}$$

Diagram 15

$$\coprod_{\xi,\acute{\xi}} \mathbf{X}(\xi) \times Hom(\acute{\xi}, \xi) \times \mathbf{A}(\acute{\xi}) \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \coprod_{\xi} \mathbf{X}(\xi) \times \mathbf{A}(\xi) \xrightarrow{\chi} \mathbf{X} \otimes_{\mathcal{O}_B} \mathbf{A}$$

Diagram 16

Furthermore if we define the arrows

$$k_\xi : \mathbf{X} \otimes_{\mathcal{O}_B} \mathbf{A} \longrightarrow \Xi, \qquad l_\xi : \mathbf{X}(\xi) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$$

they are related under the fundamental adjunction by

$$k_\xi(p, q) = l_\xi(p)(q), \qquad \xi \in \mathcal{O}_B,\ p \in \mathbf{X}(\xi), q \in \mathbf{A}(\xi)$$

Here we consider k as a function on $\coprod_\xi \mathbf{X}(\xi) \times \mathbf{A}(\xi)$ with components $k_\xi : \mathbf{X}(\xi) \times \mathbf{A}(\xi) \longrightarrow \Xi$ satisfying

$$k_{\xi'}(pv, q) = k_\xi(p, vq)$$

in agreement with the equivalence relation defined above.

Now we replace the category **Sets** by the category of quantum observables $\mathcal{O}_Q$ under study. The element q in the set $\mathbf{A}(\xi)$ is replaced by a generalized element $q : \mathbf{A}(\zeta) \to \mathbf{A}(\xi)$ from some modelling object $\mathbf{A}(\zeta)$ of $\mathcal{O}_Q$. Then we consider k as a function $\coprod_{(\xi,p)} \mathbf{A}(\xi) \longrightarrow \Xi$ with components $k_{(\xi,p)} : \mathbf{A}(\xi) \to \Xi$ for each $p \in \mathbf{X}(\xi)$, that for all arrows $v : \xi' \longrightarrow \xi$ satisfy

$$k_{(\xi', pv)} = k_{(\xi,p)} \circ \mathbf{A}(v)$$

Then the condition defining the bijection holding by virtue of the fundamental adjunction is given by

$$k_{(\xi,p)} \circ q = l_\xi(p) \circ q : \mathbf{A}(\zeta) \to \Xi$$

This argument, being natural in the object $\mathbf{A}(\zeta)$, is determined by setting $\mathbf{A}(\zeta) = \mathbf{A}(\xi)$ with q being the identity map. Hence the bijection takes the form $k_{(\xi,p)} = l_\xi(p)$, where $k : \coprod_{(\xi,p)} \mathbf{A}(\xi) \longrightarrow \Xi$, and $l_\xi : \mathbf{X}(\xi) \to \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\xi), \Xi)$.

6. SYSTEM OF MEASUREMENT LOCALIZATIONS FOR QUANTUM OBSERVABLES

The conceptual basis underlying the notion of a system of localizations for a quantum observable, which will be defined subsequently, is an implication of the categorical principle according to which, the quantum object Ξ in $\mathcal{O}_Q$ is possible to be comprehended by means of certain structure preserving maps $\xi \longrightarrow \Xi$ with local or modelling objects Boolean observables ξ in $\mathcal{O}_B$ as their domains. It is obvious, that any single map from any modelling Boolean observable to a quantum observable, is not adequate to determine it entirely, and hence, it contains only a fraction of the total information content included in it. This problem may be tackled, only if, we employ many appropriate structure preserving maps from the modelling Boolean observables to a quantum observable simultaneously, so as to cover it completely. In turn the information available about each map of the specified kind may be used to determine the quantum observable itself. In this case we conceive the family of such maps as the generator of a system of localizations for a quantum observable. The notion of local is characterized using a notion of topology on $\mathcal{O}_B$, the axioms of which express closure conditions on the collection of modelling objects.

6.1. The Notion of Grothendieck Topology on $\mathcal{O}_B$

We start our discussion by explicating the notion of a topology on the category of Boolean observables $\mathcal{O}_B$. A topology on $\mathcal{O}_B$ is a system of arrows $\mathbf{\Lambda}$, where for each object ξ there is a set $\mathbf{\Lambda}(\xi)$ that contains indexed families of $\mathcal{O}_B$-morphisms,

$$\mathbf{\Lambda}(\xi) = \{\psi_i : \xi_{-i} \to \xi, i \in I\}$$

that is, Boolean homomorphisms to ξ, such that certain appropriate conditions are satisfied.

The notion of a topology on the category of Boolean observables $\mathcal{O}_B$ is a categorical generalization of a system of set-theoretical covers on a topology $\mathbf{T}$, where a cover for $U \in \mathbf{T}$ is a set $\{U_i : U_i \in \mathbf{T}, \mathbf{i} \in \mathbf{I}\}$ such that $\cup U_i = U$. The generalization is achieved by noting that the topology ordered by inclusion is a poset category and that any cover corresponds to a collection of inclusion arrows $U_i \to U$. Given this fact, any family of arrows contained in $\mathbf{\Lambda}(\xi)$ of a topology is a cover as well.

The specification of a categorial or Grothendieck topology on the category of Boolean observables takes place through the introduction of appropriate covering devices, called covering sieves. For an object ξ in $\mathcal{O}_B$, a ξ-sieve is a family R of $\mathcal{O}_B$-morphisms with codomain ξ, such that if $\zeta \to \xi$ belongs to R and $\vartheta \to \zeta$ is any $\mathcal{O}_B$-morphism, then the composite $\vartheta \to \zeta \to \xi$ belongs to R.

A Grothendieck topology on the category of Boolean observables $\mathcal{O}_B$, is a system J of sets, $J(\xi)$ for each ξ in $\mathcal{O}_B$, where each $J(\xi)$ consists of a set of ξ-sieves, (called the covering sieves), that satisfy the following conditions:

1. For any ξ in $\mathcal{O}_B$ the maximal sieve $\{g : \mathrm{cod}(g) = \xi\}$ belongs to $J(\xi)$ (maximality condition).
2. If R belongs to $J(\xi)$ and $f : \zeta \to \xi$ is a $\mathcal{O}_B$-morphism, then $f^*(R) = \{h : \zeta \to \xi, f \cdot h \in R\}$ belongs to $J(\zeta)$ (stability condition).
3. If R belongs to $J(\xi)$ and S is a sieve on ζ, where for each $f : \zeta \to \xi$ belonging to R, we have $f^*(S)$ in $J(\zeta)$, then S belongs to $J(\xi)$ (transitivity condition).

The small category $\mathcal{O}_B$ together with a Grothendieck topology $\mathbf{J}$, is called a Boolean observables site.

6.2. The Grothendieck Topology of Epimorphic Families

We consider $\mathcal{O}_B$ as a model category, whose set of objects $\{\xi_i : i \in I\}$, I: index set, generate $\mathcal{O}_Q$, in the sense that,

$$\mathbf{A}(\xi_i) \xrightarrow{w} \Xi \underset{u}{\overset{v}{\rightrightarrows}} \Gamma$$

the identity $w \cdot v = w \cdot u$, for every arrow $w : \mathbf{A}(\xi_i) \to \Xi$, and every ξ_i, implies that $v = u$. Equivalently we can say that the set of all arrows $w : \mathbf{A}(\xi_i) \to \Xi$, constitute an epimorphic family.

The consideration that $\mathcal{O}_B$ is a generating model category of $\mathcal{O}_Q$ points exactly to the depiction of the appropriate Grothendieck topology on $\mathcal{O}_B$.

We assert that a sieve S on a Boolean observable ξ in $\mathcal{O}_B$ is to be a covering sieve of ξ, when the arrows $s : \zeta \to \xi$ belonging to the sieve S together form an epimorphic family in $\mathcal{O}_Q$. This requirement may be equivalently expressed in terms of a map

$$\Phi_S : \coprod_{(s:\zeta\to\xi)\in S} \zeta \to \xi$$

being an epi in $\mathcal{O}_Q$.

We will show that the choice of covering sieves on Boolean observables ξ in $\mathcal{O}_B$, as being epimorphic families in $\mathcal{O}_Q$, does indeed define a Grothendieck topology on $\mathcal{O}_B$.

First of all we notice that the maximal sieve on each Boolean observable ξ, includes the identity $\xi \to \xi$, thus it is a covering sieve. Next, the transitivity property of the depicted covering sieves is obvious. It remains to demonstrate that the covering sieves remain stable under pullback. For this purpose we consider the pullback of such a covering sieve S on ξ along any arrow $h : \xi' \to \xi$ in $\mathcal{O}_B$

$$\begin{array}{ccc} \coprod_{s\in S} \zeta \times_\xi \xi' & \longrightarrow & \xi' \\ \downarrow & & \downarrow h \\ \coprod_{s\in S} \zeta & \xrightarrow{\Phi} & \xi \end{array}$$

The Boolean observables ξ in $\mathcal{O}_B$ generate the category of quantum observables $\mathcal{O}_Q$, hence, there exists for each arrow $s : \vartheta \to \xi$ in S, an epimorphic family of arrows $\coprod[\xi]^s \to \vartheta \times_\xi \xi'$, or equivalently $\{[\xi]^s_j \to \vartheta \times_\xi \xi'\}_j$, with each domain $[\xi]^s$ a Boolean observable.

Consequently the collection of all the composites:

$$[\xi]^s_j \to \vartheta \times_\xi \xi' \to \xi'$$

for all $s : \vartheta \to \xi$ in S, and all indices j together form an epimorphic family in $\mathcal{O}_Q$, that is contained in the sieve $h^*(S)$, being the pullback of S along $h : \xi \to \xi'$. Therefore the sieve $h^*(S)$ is a covering sieve.

6.3. Covering Sieves as Localization Systems

If we consider a quantum observable Ξ, and all quantum algebraic homomorphisms of the form $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$, with domains ξ, in the generating model category of Boolean observables $\mathcal{O}_B$, then the family of all these maps ψ_ξ, constitute an epimorphism:

$$S : \coprod_{(\xi \in \mathcal{O}_B, \psi_\xi : \mathbf{A}(\xi) \to \Xi)} \mathbf{A}(\xi) \to \Xi$$

We say that a sieve on a quantum observable defines a covering sieve by objects of its generating model category $\mathcal{O}_B$, when the quantum algebraic homomorphisms belonging to the sieve define the preceding epimorphism.

From the physical point of view covering sieves by Boolean observables, are equivalent with Boolean localization systems of quantum observables. These localization systems filter the information of the quantum kind of observable structure through Boolean domains, associated with procedures of measurement of observables. We will discuss localizations systems in detail, in order to unravel the physical meaning of the requirements underlying the notion of Grothendieck topology, and subsequently, the notion of covering sieves defined previously. It is instructive to begin with the notion of a system of prelocalizations for a quantum observable.

A system of prelocalizations for quantum observable Ξ in $\mathcal{O}_Q$ is a subfunctor of the Hom-functor $\mathbf{R}(\Xi)$ of the form $\mathbf{S} : \mathcal{O}_B^{op} \to$ **Sets**, namely for all ξ in $\mathcal{O}_B$ it satisfies $\mathbf{S}(\xi) \subseteq [\mathbf{R}(\Xi)](\xi)$. Hence a system of prelocalizations for quantum observable Ξ in $\mathcal{O}_Q$ is an ideal $\mathbf{S}(\xi)$ of quantum algebraic homomorphisms of the form

$$\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi, \quad \xi \in \mathcal{O}_B$$

such that $\{\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$ in $\mathbf{S}(\xi)$, and $\mathbf{A}(v) : \mathbf{A}(\xi') \to \mathbf{A}(\xi)$ in $\mathcal{O}_Q$ for $v : \xi' \to \xi$ in $\mathcal{O}_B$, implies $\psi_\xi \circ \mathbf{A}(v) : \mathbf{A}(\xi') \longrightarrow \mathcal{O}_Q$ in $\mathbf{S}(\xi)\}$.

The introduction of the notion of a system of prelocalizations is forced on the basis of operational physical arguments. According to Kochen–Specker theorem it is not possible to understand completely a quantum mechanical system with the use of a single system of Boolean devices. On the other side, in every concrete experimental context, the set of events that have been actualized in this context forms a Boolean algebra. In the light of this we can say that any Boolean domain object $(B_\Xi, [\psi_B]_\Xi : \mathbf{A}(B_\Xi) \longrightarrow L)$ in a system of prelocalizations for quantum event algebra, making Diagram 17 commutative, corresponds to a set of Boolean classical events that become actualized in the experimental context of B. These Boolean objects play the role of localizing devices in a quantum event structure, that are induced by measurement situations. The above observation is equivalent to the statement that a measurement-induced Boolean algebra serves as a reference frame, in a topos-theoretical environment, relative to which a measurement result is being coordinatized. Correspondingly, by Diagram 17, we obtain naturally the

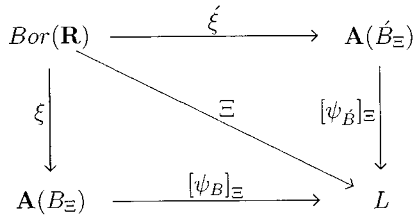

Diagram 17

notion of coordinatizing Boolean observables in a system of prelocalizations for a quantum observable over quantum event algebra L. The same notion suggests an effective way of comprehending quantum theory in a contextual perspective, pointing to a relativity principle of a topos-theoretical origin. Concretely it supports the assertion that the quantum world is the universe of varying Boolean reference frames, which interconnect to form a coherent picture in a nontrivial way.

Adopting the aforementioned perspective on quantum observable structures, the operation of the Hom-functor $\mathbf{R}(\Xi)$ is equivalent to depicting an ideal of algebraic homomorphisms which are to play the role of local coverings of a quantum observable by modelling objects. The notion of a system of prelocalizations formalizes an intuitive idea, according to which, if we sent many coordinatizing Boolean observables into the quantum observable homomorphically, then we would expect these modelling objects would prove to be enough for the complete determination of the quantum observable. If we consider a geometrical viewpoint, we may legitimately characterize metaphorically the maps $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$, $\xi \in \mathcal{O}_B$, in a system of prelocalizations for quantum observable Ξ as Boolean observable charts. Correspondingly the modelling Boolean domain objects $(B_\Xi, [\psi_B]_\Xi : \mathbf{A}(B_\Xi) \longrightarrow L)$ in a system of prelocalizations for a quantum event algebra, making Diagram 17 commutative, may be characterized as measurement charts. Subsequently, their domains B_Ξ may be called Boolean coefficient domains induced by measurement, the elements of B_Ξ measured local Boolean coefficients, and the elements of L quantum events, (or quantum propositins in a logical interpretation), coordinatized by Boolean coefficients. Finally, the Boolean homomorphisms $v : B_\Xi \longrightarrow B'_\Xi$ in $\mathcal{B}$ play the equivalent role of transition maps.

Under these intuitive identifications, we say that a family of Boolean observable charts $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$, $\xi \in \mathcal{O}_B$ (or correspondingly a family of Boolean measurement charts $[\psi_B]_\Xi : \mathbf{A}(B_\Xi) \longrightarrow L$ making Diagram 17 commutative), is the generator of the system of prelocalization **S** iff this system is the smallest among all that contains that family. It is evident that a quantum observable, and correspondingly the quantum event algebra over which it is defined, can have many systems of measurement prelocalizations, that, remarkably, form an ordered structure. More specifically, systems of prelocalization constitute a partially-ordered set

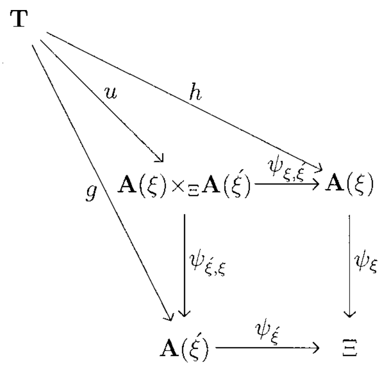

Diagram 18

under inclusion. Furthermore, the intersection of any number of systems of prelocalization is again a system of prelocalization. We emphasize that the minimal system is the empty one, namely $\mathbf{S}(\xi) = \emptyset$ for all $\xi \in \mathcal{O}_B$, whereas the maximal system is the Hom-functor $\mathbf{R}(\Xi)$ itself, or equivalently, all quantum algebraic homomorphisms $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$.

The transition from a system of prelocalizations to a system of localizations for a quantum observable, can be effected under the restriction that, certain compatibility conditions have to be satisfied on the overlap of the modelling Boolean charts covering the quantum observable under investigation. In order to accomplish this we use a pullback diagram in $\mathcal{O}_Q$ Diagram 18.

The pullback of the Boolean charts $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$, $\xi \in \mathcal{O}_B$ and $\psi_{\xi'} : \mathbf{A}(\xi') \longrightarrow \Xi$, $\xi' \in \mathcal{O}_B$ with common codomain the quantum observable Ξ, consists of the object $\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\xi')$ and two arrows $\psi_{\xi\xi'}$ and $\psi_{\xi'\xi}$, called projections, as shown in the above diagram. The square commutes and for any object T and arrows h and g that make the outer square commute, there is a unique $u : T \longrightarrow \mathbf{A}(\xi) \times_\Xi \mathbf{A}(\xi')$ that makes the whole diagram commutative. Hence we obtain the condition:

$$\psi_{\xi'} \circ g = \psi_\xi \circ h$$

The pullback of the Boolean observable charts $\psi_\xi : \mathbf{A}(\xi) \longrightarrow \Xi$, $\xi \in \mathcal{O}_B$, and $\psi_{\xi'} : \mathbf{A}(\xi') \longrightarrow \Xi$, $\xi' \in \mathcal{O}_B$, is equivalently characterized as their fiber product, because $\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\xi')$ is not the whole product $\mathbf{A}(\xi) \times \mathbf{A}(\xi')$ but the product taken fiber by fiber. We notice that if ψ_ξ and $\psi_{\xi'}$ are 1-1, then their pullback is isomorphic with the intersection $\mathbf{A}(\xi) \cap \mathbf{A}(\xi')$. Then we can define the pasting map, which is an isomorphism, as follows:

$$\Omega_{\xi,\xi'} : \psi_{\xi'\xi}(\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\xi')) \longrightarrow \psi_{\xi\xi'}(\mathbf{A}(\xi) \times_\Xi \mathbf{A}(\xi'))$$

by putting

$$\Omega_{\xi,\xi'} = \psi_{\xi\xi'} \circ \psi_{\xi'\xi}^{-1}$$

Then we have the following conditions:

$$\Omega_{\xi,\xi} = 1_{\xi} \qquad 1_{\xi} := id_{\xi}$$

$$\Omega_{\xi,\xi'} \circ \Omega_{\xi',\xi''} = \Omega_{\xi,\xi''} \quad \text{if} \quad \mathbf{A}(\xi) \cap \mathbf{A}(\xi') \cap \mathbf{A}(\xi'') \neq 0$$

$$\Omega_{\xi,\xi'} = \Omega_{\xi',\xi} \quad \text{if} \quad \mathbf{A}(\xi) \cap \mathbf{A}(\xi') \neq 0$$

The pasting map assures that $\psi_{\xi'\xi}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\xi'))$ and $\psi_{\xi\xi'}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\xi'))$ are going to cover the same part of the quantum observable in a compatible way.

It is obvious that the above compatibility conditions are translated immediately to corresponding compatibility conditions concerning Boolean measurement charts on the quantum event structure.

Given a system of measurement prelocalizations for quantum observable $\Xi \in \mathcal{O}_Q$, and correspondingly for the quantum event algebra over which it is defined, we call it a *system of localizations* iff the above compatibility conditions are satisfied and moreover the quantum algebraic structure is preserved.

We assert that the above compatibility conditions provide the necessary relations for understanding a system of measurement localizations for a quantum observable as a structure sheaf or sheaf of Boolean coefficients consisting of local Boolean observables. This is connected to the fact that systems of measurement localizations are actually subfunctors of the representable Hom-functor $\mathbf{R}(\Xi)$ of the form $\mathbf{S} : \mathcal{O}_B^{op} \to \mathbf{Sets}$, namely for all ξ in $\mathcal{O}_B$ satisfy $\mathbf{S}(\xi) \subseteq [\mathbf{R}(\Xi)](\xi)$. In this sense the pullback compatibility conditions express gluing relations on overlaps of Boolean observable charts and convert a presheaf subfunctor of the Hom-functor into a sheaf for the Grothendieck topology specified.

The concept of sheaf expresses exactly the pasting conditions that local modelling objects have to satisfy, namely, the way by which local data can be collated. We stress the point that the transition from locally defined properties to global consequences happens via a compatible family of elements over a cover of the complex object. A cover, or equivalently a localization system of the global, complex object, being a quantum observable structure in the present scheme, can be viewed as providing a decomposition of that object into simpler modelling objects.

The comprehension of a measurement localization system as a sheaf of Boolean coefficients permits the conception of a quantum observable (or of its associated quantum event algebra) as a generalized manifold, obtained by pasting the $\psi_{\xi'\xi}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\xi'))$ and $\psi_{\xi\xi'}(\mathbf{A}(\xi) \times_{\Xi} \mathbf{A}(\xi'))$ covers together by the transition functions $\Omega_{\xi,\xi'}$.

More specifically, the equivalence relations in the category of elements of such a structure sheaf, represented by a Boolean system of measurement localizations, have to be taken into account according to the analysis of the fundamental

Hom-tensor adjunction of Section 5. Equivalence relations of this form give rise to congruences in the structure sheaf of Boolean coefficients, which are expressed categorically as a colimit in the category of elements of such a structure sheaf. In this perspective the generalized manifold, which represents categorically a quantum observable object, is understood as a colimit in a sheaf of Boolean coefficients, that contains compatible families of modeling Boolean observables. It is important to underline the fact that the organization of Boolean coordinatizing objects in localization systems takes the form of interconnection of these modeling objects through the categorcal construction of colimit, the latter being the means to comprehend an object of complex structure (Quantum Observable) from simpler objects (Boolean Observables).

The above ideas provide the basis for the formulation of a representation theorem concerning quantum observables and their associated quantum event algebras as we shall present in the following section.

7. REPRESENTATION OF QUANTUM OBSERVABLES AND EVENT ALGEBRAS

7.1. Unit and Counit of the Fundamental Adjunction

We focus again our attention in the fundamental adjunction and investigate the unit and the counit of it. For any presheaf $\mathbf{X}$ in the topos $\mathbf{Sets}^{\mathcal{O}_B^{op}}$, the **unit** $\delta_{\mathbf{X}} : \mathbf{X} \longrightarrow \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\_), \mathbf{X} \otimes_{\mathcal{O}_B} \mathbf{A})$ has components:

$$\delta_{\mathbf{X}}(\xi) : \mathbf{X}(\xi) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\xi), \mathbf{X} \otimes_{\mathcal{O}_B} \mathbf{A})$$

for each Boolean observable object ξ of $\mathcal{O}_B$.

If we make use of the representable presheaf $y[\xi]$ we obtain

$$\delta_{\mathbf{y}[\xi]} : \mathbf{y}[\xi] \to \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\_), \mathbf{y}[\xi] \otimes_{\mathcal{O}_B} \mathbf{A})$$

Hence for each object ξ of $\mathcal{O}_B$ the unit, in the case considered, corresponds to a map

$$\mathbf{A}(\xi) \to \mathbf{y}[\xi] \otimes_{\mathcal{O}_B} \mathbf{A}$$

But since

$$\mathbf{y}[\xi] \otimes_{\mathcal{O}_B} \mathbf{A} \cong \mathbf{A}(\xi)$$

the unit for the representable presheaf of Boolean observables is clearly an isomorphism. By the preceding discussion we can see that Diagram 19 commutes.

Thus the unit of the fundamental adjunction referring to the representable presheaf of the category of Boolean observables provides a quantum algebraic homomorphism, $\mathbf{A}(\xi) \longrightarrow \mathbf{y}[\xi] \otimes_{\mathcal{O}_B} \mathbf{A}$, which is an isomorphism.

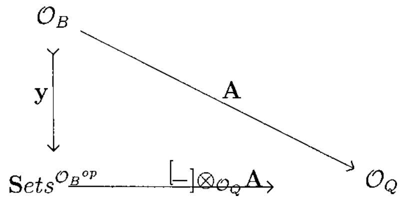

Diagram 19

On the other side, for each quantum observable object Ξ of $\mathcal{O}_Q$ the *counit* is

$$\epsilon_\Xi : \mathrm{Hom}_{\mathcal{O}_Q}(\mathbf{A}(\_), \Xi) \otimes_{\mathcal{O}_B} \mathbf{A} \longrightarrow \Xi$$

The counit corresponds to the vertical map in Diagram 20.

7.2. Boolean Manifold Representation by Measurement Localizations

The manifold representation of a quantum observable structure in terms of Boolean measurement localizations, consisting of Boolean reference frames in a topos-theoretical environment, is described by the following proposition:

Proposition. *Given a quantum observable Ξ in $\mathcal{O}_Q$ and a system of compatible measurement prelocalizations consisting of Boolean observables, then it is a system of measurement localizations iff the counit of the fundamental adjunction restricted to this system is an isomorphism. This statement may equivalently and more fundamentally be expressed in terms of the quantum event algebra over which observables are defined, if we take into account Diagram 17, as follows*

Proposition. *Given a quantum event algebra L in $\mathcal{L}$ and a system of compatible measurement prelocalizations for quantum observable Ξ over L, consisting of Boolean measurement charts, then it is a system of measurement localizations, iff the counit of the fundamental adjunction restricted to this system is an isomorphism.*

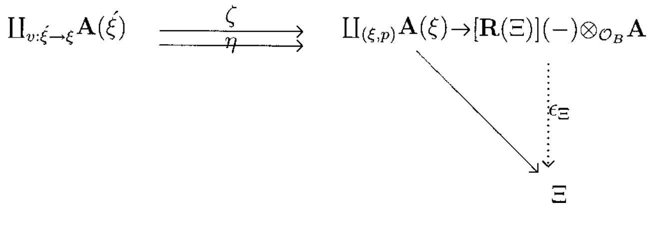

Diagram 20

In this case we say that a quantum event algebra L in $\mathcal{L}$ admits a Boolean manifold representation induced by Boolean measurement charts for observables defined over L.

Proof: The proof of the proposition goes as follows: (For simplicity in the notation we avoid writing the observable index Ξ explicitly when we refer to measurement charts).

We suppose that we are given a quantum event algebra, a system of measurement compatible prelocalizations of it, and moreover let the counit of the adjunction (expessed in terms of event algebras) restricted to this system is an isomorphism

$$\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{A} \longrightarrow L$$

such that

$$\psi_B = \epsilon_L \circ [\psi_B \otimes \_]$$

or in the notation of elements equivalently:

$$\epsilon_L([\psi_B \otimes a]) = \psi_B(a), \quad a \in \mathbf{A}(B)$$

where $\psi_B(a) = \Xi(\Xi_B^{-1}(a))$, for all $\psi_B : \mathbf{A}(B) \longrightarrow L$ according to the commutative triangle Diagram 21.

Let $\mathbf{T}$ be any system of measurement prelocalizations for quantum event algebra L in $\mathcal{L}$. Since the counit ϵ_L is surjective map, for given element l in L we obtain $l = \psi_B(a) = \epsilon_L([\psi_B \otimes a])$ for some $\psi_B : \mathbf{A}(B) \longrightarrow L$. Since $\mathbf{T}$ is a system of prelocalizations, we have $\psi_B = \psi_C \circ \mathbf{A}(v)$ for some $v : C \longrightarrow B$ in $\mathcal{B}$ and ψ_C in $\mathbf{T}$. Hence

$$l = \psi_C \circ [\mathbf{A}(v)](a) = \psi_C(b), \quad \psi_C : \mathbf{A}(C) \longrightarrow L \in \mathbf{T}, b \in \mathbf{A}(C)$$

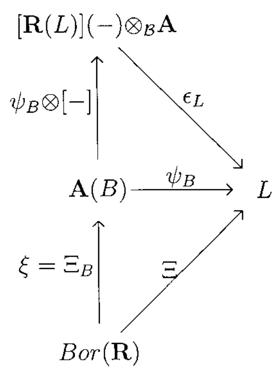

Diagram 21

Hence for every quantum event l there exists a measurement Boolean chart in $\mathbf{T}$ and Boolean coefficients a in $\mathbf{A}(B)$ such that $l = \psi_B(a)$, or else every quantum event gets covered.

Moreover let $\psi_B : \mathbf{A}(B) \longrightarrow L$ and $\psi_C : \mathbf{A}(C) \longrightarrow L$ in $\mathbf{T}$. Then the fiber product structure $\mathbf{K} = \mathbf{A}(B) \times_L \mathbf{A}(C)$ with projections $h : \mathbf{K} \longrightarrow \mathbf{A}(B)$, $g : \mathbf{K} \longrightarrow \mathbf{A}(C)$, and the fact that the counit is 1-1, provides compatibility relations on overlaps for the Boolean coordinates of a quantum event. Concretely, for every two Boolean charts $\psi_B : \mathbf{A}(B) \longrightarrow L$ and $\psi_C : \mathbf{A}(C) \longrightarrow L$ in $\mathbf{T}$ such that $\psi_B(a) = \psi_C(b)$, $a \in \mathbf{A}(B)$ and $b \in \mathbf{A}(C)$, there exists a pair of transition functions provided by the projections of the fiber product, $h : \mathbf{K} \longrightarrow \mathbf{A}(B)$, $g : \mathbf{K} \longrightarrow \mathbf{A}(C)$ and a Boolean coefficient $k \in \mathbf{K}$ such that

$$\psi_B \circ h = \psi_C \circ g, \qquad a = h(k), b = g(k)$$

Furthermore from the definition of the left adjoint functor we know that $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{A}$ has a quantum event algebra structure. Since the counit is an isomorphism, the quantum event algebra structure is into 1-1 correspondence with that of L. More explicitly the quantum event algebra structure of L is identical with the one induced by the system $\mathbf{T}$, namely:

$$l = 1 \Leftrightarrow l = \psi_B(1), \forall \psi_B \in \mathbf{T}$$

$$l = m^* \Leftrightarrow [m = \psi_B(a) \Rightarrow l = \psi_B(a^*)], \forall \psi_B \in \mathbf{T}, \forall a \in \mathrm{Dom}(\psi_B)$$

$$l \leq m \Leftrightarrow [l = \psi_B(a) \wedge m = \psi_B(b)] \Rightarrow a \leq b, \forall \psi_B \in \mathbf{T}, \forall a, b \in \mathrm{Dom}(\psi_B)$$

Conversely, let $\mathbf{T}$ be any system of measurement localizations for quantum event algebra L in $\mathcal{L}$, such that the measurement Boolean charts are endowed with the properties of covering entirely L, are compatible on overlaps, and carry a quantum event algebra structure. Then we claim that the counit $\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{A} \longrightarrow L$ defines a quantum algebraic homomorphism which is an isomorphism.

Firstly, by the property of covering, the counit has to be surjective. In order to prove that it is 1-1, we suppose that $\psi_B : \mathbf{A}(B) \longrightarrow L$ and $\psi_C : \mathbf{A}(C) \longrightarrow L$ in $\mathbf{T}$ are in a system of localizations, and let $\epsilon_L([\psi_B \otimes a]) = \epsilon_L([\psi_C \otimes b])$, or equivalently $\psi_B(a) = \psi_C(b)$. We wish to show that $[\psi_B \otimes a] = [\psi_C \otimes b]$. We set $\psi_B = \psi_D \circ \mathbf{A}(v)$ and $\psi_C = \psi_E \circ \mathbf{A}(w)$ for ψ_D and ψ_E in $\mathbf{T}$, and some transition functions $\mathbf{A}(v)$, $\mathbf{A}(w)$. It is clear that $[\phi_B \otimes a] = [\psi_D \otimes [\mathbf{A}(v)](a)]$ and $[\psi_C \otimes b] = [\psi_E \otimes [\mathbf{A}(w)](b)]$ and moreover $\psi_D([\mathbf{A}(v)](a)) = \psi_E([\mathbf{A}(w)](b))$. At this point the compatibility on overlaps property of the system of localizations considered, will supply us with transition functions $\mathbf{A}(v')$, $\mathbf{A}(w')$, such that we are going to have $[\psi_D \otimes [\mathbf{A}(v)](a)] = [\psi_E \otimes [\mathbf{A}(w)](b)]$.

It remains to show that the counit preserves the quantum algebraic structure in order to establish the isomorphism. Immediately we can show that

$$\epsilon_L([\psi_B \otimes 1]) = \psi_B(1) = 1$$

$$\epsilon_L(([\psi_B \otimes a])^*) = \epsilon_L([\psi_B \otimes a^*]) = \psi_B(a^*) = [\psi_B(a)]^* = [\epsilon_L([\psi_B \otimes a])]^*$$

The partial ordering can be shown as follows: $[\psi_B \otimes a] \leq [\psi_C \otimes b]$ iff $\langle c \leq d \Rightarrow \psi_K(c) \leq \psi_K(d)\rangle$ or equivalently $\epsilon_L([\psi_K \otimes c]) \leq \epsilon_L([\psi_K \otimes d])$, where $[\psi_B \otimes a] = [\psi_K \otimes c]$ and $[\psi_C \otimes b] = [\psi_K \otimes c]$, and the pullback of the arrows ψ_B and ψ_C has been used, in which $h(c) = a$, $g(d) = b$, $\psi_K = \psi_B \circ h = \psi_B \circ g$. Next we observe that since the counit is onto and 1-1, we obtain: $\epsilon_L([\psi_B \otimes a]) \leq \epsilon_L([\psi_C \otimes b])$ iff $\epsilon_L([\psi_K \otimes c]) \leq \epsilon_L([\psi_K \otimes d]) \Rightarrow \psi_K(c) \leq \psi_K(d) \Rightarrow \langle c \leq d$ iff $[\psi_B \otimes a] \leq [\psi_C \otimes b]\rangle$. □

8. SEMANTICAL ASPECTS OF BOOLEAN LOCALIZATION SYSTEMS

By virtue of the fundamental proposition we conclude that:

1. A system of localizations $\mathbf{S} : \mathcal{B}^{op} \longrightarrow$ **Sets** plays the role of a measurement atlas for a quantum event algebra L in $\mathcal{L}$.
2. The quantum event algebra L, endowed with an atlas of Boolean measurement localizations, is a Boolean manifold.
3. The objects of the category of elements $\mathbf{G}(\mathbf{R}(L), B)$ are the local modeling measurement Boolean charts and its maps are the pasting maps. These objects are identified as the reference frames on a quantum observable structure, considered in a topos-theoretical environment, in conjunction with the adjunction eastablished between the Boolean and quantum species of observable structure.

The surjective property of the counit implies that the Boolean charts in $\mathbf{G}(\mathbf{R}(L), B)$ cover entirely the quantum event algebra L, whereas its injective property implies that any two measurement Boolean charts are compatible. Moreover since the counit is also an algebraic homomorphism, it preserves the structure, hence in effect, the quantum event algebra L is determined completely by the Boolean measurement charts and their compatibility relations in a system of localizations of it. Each chart corresponds to a set of Boolean events actualized locally in a measurement situation. The equivalence classes of measurement charts represent quantum events in L, through compatible coordinatizations by Boolean coefficients. We notice that, since two different local Boolean measurement charts may overlap, there exists the possibility of probing the quantum structure by observing quantum events from different frames, or in different contexts. But due to the presence of the equivalence and compatibility relations, these different contexts of observing are equivalent and moreover establish the same quantum event.

The interpretation of the quantum observable structure via Boolean reference frames, has been based on the use of observable coordinatizing objects, belonging to the Boolean species, as local modeling figures for probing the objects belonging to the quantum species of structure. The Boolean objects give rise to structure preserving maps, having the modeling objects themselves as their domains, which by fitting in systems of compatible localizations together, provide an isomorphism between quantum event algebras and Boolean measurement localization systems. Consequently, the structure of a quantum event algebra is being generated by the information that its structure preserving maps, encoded as Boolean measurement charts in localization systems, carry, as well as their compatibility relations. This process leads naturally to a contextual description of quantum events (or quantum propositions) with respect to Boolean reference frames of measurement, and finally to a representation of them as equivalence classes of unsharp Boolean events. Equivalently, quantum observables are being understood through isomorphism classes of their Boolean localizations on measurement charts.

The conceptual basis of the proposed relativistic perspective on quantum structure, established by systems of Boolean measurement localization systems, is located on the physical meaning of the adjunction between presheaves of Boolean observables and quantum observables, and the subsequent categorical equivalence provided by the Boolean manifold picture.

Let us consider that $\mathbf{Sets}^{\mathcal{B}^{op}}$ is the universe of Boolean observable event structures modelled in **Sets** by observers, or else the world of Boolean windows, and $\mathcal{L}$ that of quantum event structures. In the proposed interpretation the functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \longrightarrow \mathcal{L}$ can be comprehended as a translational code from Boolean windows to the quantum species of event structure, whereas the functor $\mathbf{R} : \mathcal{L} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ as a translational code in the inverse direction. In general, the content of the information is not possible to remain completely invariant translating from one language to another and back. However, there remain two ways for a Boolean-event algebra variable set **P**, or else Boolean window to communicate a message to a quantum event algebra L. Either the information is given in quantum terms with **P** translating, which we can be represented as the quantum homomorphism $\mathbf{LP} \longrightarrow L$, or the information is given in Boolean terms with L translating, that, in turn, can be represented as the natural transformation $\mathbf{P} \longrightarrow \mathbf{R}(L)$. In the first case, from the perspective of L information is being received in quantum terms, while in the second, from the perspective of **P** information is being sent in Boolean terms. The natural bijection then corresponds to the assertion that these two distinct ways of communicating are equivalent. In this mode of arguing, the left adjunction operator can be characterized as the *Quantization functor*, whereas the right adjunction operator as the *Classicalization functor*. Consequently, the fact that these two functors are adjoint to each other, expresses an amphidromous dependent variation, concerning the crystallization of the meaning of the information related to observation. Equivalently stated, the adjunctive correspondence essentially relates relations and provides the necessary and sufficient constraints for establishing a notion of mutually dependent variation, in the interpretation of the information content shared by the Boolean and quantum species of observable structure. At a further stage, the representation of a quantum observable as a categorical colimit, resulting from the same adjunctive relation, reveals an entity that can admit a multitude of instantiations in Boolean localization systems. The informational content of all different instantiations remains invariant if and only if the counit of the adjunction restricted to Boolean localization systems is an isomorhism, and moreover, is equivalent to the whole informational content captured by the quantum structure itself. Thus, finally, the counit isomorphism, provides a categorical equivalence signifying an invariance in the translational code of communication between Boolean windows and quantum systems.

9. BOOLEAN TRUTH VALUES AND QUANTUM LOGIC

The fibration induced by a presheaf of Boolean algebras **P**, provides the category of elements of **P**, denoted by $\mathbf{G}(\mathbf{P}, \mathcal{B})$. Its objects are all pairs (B, p), and its morphisms $(B', p') \longrightarrow (B, p)$ are those morphisms $u : B' \longrightarrow B$ of $\mathcal{B}$ for which $pu = p'$. Projection on the second coordinate of $\mathbf{G}(\mathbf{P}, \mathcal{B})$, defines a functor $\mathbf{G}_\mathbf{P} : \mathbf{G}(\mathbf{P}, \mathcal{B}) \longrightarrow \mathcal{B}$.

Hence $\mathbf{G}(\mathbf{P}, \mathcal{B})$ together with the projection functor $\mathbf{G}_\mathbf{P}$ constitute the split discrete fibration induced by **P**, and $\mathcal{B}$ is the base category of the fibration. We note that the fibration is descrete because the fibers are categories in which the only arrows are identity arrows. Moreover, if B is an object of $\mathcal{B}$, the inverse image under $\mathbf{G}_\mathbf{P}$ of B is simply the set **P**(B).

According to the proposed scheme of interpretation, the objects of the category of elements $\mathbf{G}(\mathbf{R}(L), B)$ constitute local measurement Boolean charts and have been identified as Boolean reference frames on a quantum observable structure.

We notice that the set of objects of $\mathbf{G}(\mathbf{R}(L), B)$ consists of all the elements of all the sets $\mathbf{R}(L)(B)$, and more concretely, has been constructed from the disjoint union of all the sets of the above form, by labelling the elements. The elements of this disjoint union are represented as pairs $(B, \psi_B : \mathbf{A}(B) \longrightarrow L)$ for all objects B of $\mathcal{B}$ and elements $\psi_B \in \mathbf{R}(L)(B)$.

Taking into account the projection functor, defined above, this set is actually a fibered structure. Each fiber is a set defined over a Boolean algebra relative to which a measurement result is being coordinatized. If we denote by (ψ_B, q) the elements of each fiber, with $\psi_B \in \mathbf{R}(L)(B)$ and $q \in \mathbf{A}(B)$, then the set of maps

$$(\psi_B, q) \longrightarrow q$$

can be interpreted as the Boolean power of the set

$$\Upsilon_B = \{(\psi_B, q), \psi_B \in \mathbf{R}(L)(B), q \in \mathbf{A}(B)\}$$

with respect to the underlying Boolean algebra B (Bell, 1985).

The Boolean power construction forces an interpretation of the Boolean algebra relative to which a measurement result is being coordinatized, as a domain of local truth values. Moreover the set of local measurement charts defined over B, is considered as a Boolean-valued set.

In this sense, the local coordinates corresponding to a Boolean domain of measurement, may be considered as fuzzy Boolean truth values.

We further observe that the set of objects of $\mathbf{G}(\mathbf{R}(L), \mathcal{B})$ consists of the disjoint union of all the fibers Υ_B, denoted by $\Upsilon = \coprod_B \Upsilon_B$. This set can also acquire a Boolean power interpretation as follows.

We define a binary relation on the set Υ according to:

$$(\psi_{B'}, q') \otimes (\psi_B, q) \text{ iff } \exists \eta : \psi_{B'} \longrightarrow \psi_B : \eta(q') = q, \psi_{B'} = \psi_B \circ \eta$$

It is evident that for any $\eta : B' \longrightarrow B$ we obtain: $(\psi_B \circ \eta, q') \otimes (\psi_B, \eta(q'))$. Furthermore, we require the satisfaction of the compatibility relations that are valid in a system of localizations. Then it is possible to define the Boolean power of the set Υ with respect to the maximal Boolean algebra belonging to such a compatible system of localizations. We may say that the Boolean coordinates, interpreted as fuzzy Boolean truth values, via the Boolean power construction, reflect a relation of indistinguishability due to overlapping of the corresponding measurement charts. The viewpoint of Boolean valued sets, instantiated as measurement charts in localization systems, has far reaching consequences regarding the interpretation of quantum logic, and will be discussed in detail, in a future work. At the present stage, we may say that the above analysis seems to substantiate Takeuti's and Davis's approach to the foundations of quantum logic (Davis, 1977; Takeuti, 1978), according to whom, quantization of a proposition of classical physics is equivalent to interpreting it in a Boolean extension of a set theoretical universe, where B is a complete Boolean algebra of projection operators on a Hilbert space. In the perspective of the present analysis, we may argue that the fibration technnique in the presheaf of Boolean algebras $\mathbf{G}(\mathbf{R}(L), \mathcal{B})$, provides the basis for a natural interpretation of the logic of quantum propositions in terms of fuzzy Boolean truth values assuming existence in the corresponding Boolean measurement contexts of localization systems.

10. DIFFERENTIAL GEOMETRY IN THE QUANTUM REGIME

10.1. Space Localization Systems

The Boolean manifold representation proposition permits the characterization of each quantum observable Ξ in $\mathcal{O}_Q$ as a system of compatible measurement localizations consisting of Boolean observables, provided that the counit of the fundamental adjunction restricted to this system is an isomorphism (Diagram 21).

An operational characterization of the Boolean manifold scheme, is afforded by the application of Stone's representation theorem for Boolean algebras. According to this theorem, it is legitimate to replace Boolean algebras by fields of subsets of a measurement space. If we replace each Boolean algebra B in $\mathcal{B}$ by its set-theoretical representation $[\Sigma, B_\Sigma]$, consisting of a local measurement space Σ and its local field of subsets B_Σ, it is possible to define local measurement space charts $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{A}(B_\Sigma) \longrightarrow L)$ and corresponding space localization systems for quantum observable Ξ over quantum event algebra L in $\mathcal{L}$. Again from local measurement space charts $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{A}(B_\Sigma) \longrightarrow L)$ we may form their equivalence classes, which modulo the conditions for compatibility on overlaps, will represent a single quantum event in L. Under these circumstances, we may interpret the equivalence classes of local space charts $\psi_{B_\Sigma} \otimes a$, $a \in \mathbf{A}(B_\Sigma)$ as the experimental actualization of the quantum events in L, corresponding to measurement of observables Ξ. The local measurement space charts $(B_\Sigma, \psi_{B_\Sigma} : \mathbf{A}(B_\Sigma) \longrightarrow L)$ and $(C_\Sigma, \psi_{C_\Sigma} : \mathbf{A}(C_\Sigma) \longrightarrow L)$ are compatible in a system of measurement localizations for observable Ξ defined over L, iff for some $(D_\Sigma, \psi_{D_\Sigma} : \mathbf{A}(D_\Sigma) \longrightarrow L)$ in the system of localizations, and $a \in \mathbf{A}(B_\Sigma)$, $b \in \mathbf{A}(C_\Sigma)$, $c, d \in \mathbf{A}(D_\Sigma)$, the following conditions hold:

$$\psi_{B_\Sigma} \otimes a = \psi_{D_\Sigma} \otimes c$$

$$\psi_{C_\Sigma} \otimes b = \psi_{D_\Sigma} \otimes d$$

We note that we could equivalently consider the local space as a compact Hausdorff space, the compact open subsets of which are the maximal filters or the prime ideals of the underlying Boolean algebra.

The pullback compatibility condition, which is in injective correspondence with the one in $\mathcal{L}$, since, it holds in a localization system, may be interpreted in the operational framework, as denoting that, two local space representations of a quantum observable satisfy the compatibility condition on overlapping regions, iff their associated measurements are equivalent to measurements sharing the same experimental arrangement. We also observe that the inverse of a local space representation of a quantum observable plays the role of a random variable on this local space Σ. Consequently, every quantum observable may be considered locally, as a measurable function defined over the local measurement space Σ.

Equivalently, random variables defined over local spaces provide Boolean coordinatizations for a quantum observable, and moreover, satisfy compatibility conditions on the overlaps of their local domains of definition. Subsequently, if we consider the category of these local spaces, the collection of measurable functions defined locally over them, provide a sheaf of Boolean co-efficients, with respect to the specification of the corresponding Grothendieck topology of epimorphic families, defined over the category of local spaces for the measurement of a quantum observable. In the perspective of the interpretation proposed in this work, the essence of a quantum observable is captured by a colimit in the category of elements of the sheaf of measurable functions, over the category of local spaces. The colimit in the category of elements of that sheaf, is expressed, as an equivalence relation on the collection of the locally defined Boolean coordinatizations according to the relation

$$pv \otimes q' = p \otimes vq', \quad p \in \mathbf{R}(L)(\xi), q' \in \mathbf{A}(\xi'), v : \xi' \longrightarrow \xi$$

Consequently, a quantum observable is represented by means of a quotient construction, consisting of the sheaf of measurable functions defined over the category of local spaces, modulo the ideal generated by the above equivalence relations. Furthermore, addition and multiplication over **R** may induce the structure of a sheaf of **R**-algebras (or a sheaf of rings).

10.2. Structure Sheaves of Generalized Spaces

Instead of the quotient sheaf of measurable functions defined over the category of local spaces, we could also consider the quotient sheaf of **R**-algebras of continuous or smooth functions corresponding to local coordinatizations of a quantum observable. A natural question that arises in this setting, is if it could be possible, to consider the above quotient sheaf of **R**-algebras as the structure algebra sheaf of a generalized space, corresponding exactly to the category of local spaces. From a physical point of view, this move would reflect the appropriate generalization of the arithmetics, or sheaves of coefficients, that have to be used in the transition from the classical to the quantum regime. In the classical case, each local space endowed with appropriate topological and differential properties, may acquire the structure of a differentiable manifold, which, in turn, is characterized completely, by the structure sheaf of smooth functions, playing the equivalent role of a Boolean observer arithmetics. Since, all classical theoretical observables are always compatible, the sheaf of coefficients (smooth functions) can be used globally, giving rise to the differential geometric mechanism of smooth manifolds.

On the other side, we have seen that a quantum observable cannot be apprehended by the use of a single Boolean observer arithmetic, but, there is a necessity of employing a whole system of local arithmetics over a category of local spaces, which are constrained to specify appropriate compatibility and equivalence relations, according to the fundamental adjunction of the categorical scheme. A single Boolean observer's arithmetic inevitably suppresses information about a quantum system and reflects a fuzzy apparatus for probing the structure of the quantum regime.

This observation is particularly enlightening when considering the notion of space, or even space–time, at the quantum regime. It naturally points to an understanding of the, so-called, space–time manifold singularities, as reflections of the inability of a Boolean observer's arithmetic, consisting of smooth functions, to probe the quantum level of structure of this entity. According to the perspective of our discussion, the appropriate generalized arithmetic that would correspond, as a structure sheaf, to a quantum conception of space–time, would be the quotient sheaf of **R**-algebras. This sheaf would contain compatible overlapping systems of local arithmetics, consisting of locally defined smooth functions over the category of local spaces, modulo the ideal corresponding to their equivalence relations, as a reflection of the interconnecting machinery of the colimit construction.

A suitable framework to accommodate structure sheaves of the above form is Abstract Differential Geometry (ADG), developed by Mallios in literature (Mallios, 1998), and discussed in relation to space–time singularities in literature (Mallios, 2002). ADG is an extension of classical Differential Geometry which does no longer use any notion of calculus. Instead of smooth functions, one starts with a general sheaf of algebras. The important thing is that these sheaves of algebras, which in the perspective of the present categorical scheme, correspond to quantum observables, can be interrelated with appropriate differentials, instantiated as suitable Leibniz sheaf morphisms, and, constituting appropriate differential complexes. Thus, sequences of quantum algebraic homomorphisms between localizations systems of quantum observables, suited to satisfy an appropriate Liebniz condition, can be qualified as chain complexes, such that the homomorphisms play exactly the role of differentials. This interpretation is suited to the development of Differential Geometry in the regime of quantum systems, from a sheaf cohomology viewpoint, and will be presented in a future work. Most significantly, it emphasizes the thesis that the intrinsic mechanism of Differential Geometry is of an operational character, referring directly to the objects of enquiry, being in the proposed scheme, the quantum observables.

11. CONCLUSIONS

The present paper proposes a relativistic perspective on quantum observable structure, established by localization systems of Boolean coordinatizing charts. According to this scheme the quantum world is comprehended via overlapping Boolean reference frames for measurement of observables, that are glued together forming a coherent structure. Most importantly, this perspective is formalized categorically, as an instance of the adjunction concept. The latter, may be further used as a formal tool for the expression of an invariant property, underlying the Boolean manifold representation. Specifically, it has been demonstrated that the physical meaning of the, adjointly related, relevant functors is associated with the operationalization of the meaning of charts, as measurement contexts, in terms of a process that has been metaphorically described as information exchange in the communication of the Boolean with the quantum level of being. In this relativistic perspective, the informational content of a quantum observable structure signifies an invariant property, with respect to Boolean domain coordinatizations, if and only if, the counit of the adjunction, restricted to covering systems, qualified as Boolean localization systems, is an isomorphism.

Finally, the semantical aspects of the categorical equivalence, obtained by the restriction of the adjunction to subfunctors of the Hom-functor, containing Boolean localization systems, point towards investigations, regarding:

On the one side, the study of consequences of a naturally forced sheaf theoretical formulation related the logic of quantum propositions, and on the other, the development of an algebraic differential geometric machinery suited to the quantum level of observable structure, as has been discussed in Sections 9 and 10. It seems that the parallel development of the above research directions, on the basis of the physical meaning of the existing categorical equivalence, will substantiate a paraphrase of Lawvere's dictum (Lawvere, 1975), according to which:

Algebraic Quantum Geometry = Geometric Quantum Logic.

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REFERENCES

Artin, M., Grothendieck, A., and Verdier, J. L. (1972). *Theorie de topos et cohomologie etale des schemas* (Springer LNM 269 and 270), Springer-Verlag, Berlin.

Bell, J. L. (1985). *Boolean-Valued Models and Independence Proofs in Set Theory*, Oxford University Press, Oxford.

Bell, J. L. (1988). *Toposes and Local Set Theories*, Oxford University Press, Oxford.

Birkhoff, G. and von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics* **37**, 823.

Butterfield, J. and Isham, C. J. (1998). A topos perspective on the Kochen–Specker theorem: I. Quantum states as generalized valuations. *International Journal of Theoretical Physics* **37**, 2669.

Butterfield, J. and Isham, C. J. (1999). A topos perspective on the Kochen–Specker theorem: II. Conceptual aspects and classical analogues. *International Journal of Theoretical Physics* **38**, 827.

Butterfield, J. and Isham, C. J. (2000). Some possible roles for topos theory in quantum theory and quantum gravity. *Foundations of Physics* **30**, 1707.

Davis, M. (1977). A relativity principle in quantum mechanics. *International Journal of Theoretical Physics* **16**, 867.

Kochen, S. and Specker E. (1967). The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics* **17**, 59.

Lawvere, F. W. (1975). Continuously variables sets: Algebraic geometry = Geometric logic. *Proceedings of the Logic Colloquium in Bristol*, North-Holland, Amsterdam.

Lawvere, F. W. and Schanuel, S. H. (1997). *Conceptual Mathematics*, Cambridge University Press, Cambridge.

MacLane, S. (1971). *Categories for the Working Mathematician*, Springer-Verlag, New York.

MacLane, S. and Moerdijk, I. (1992). *Sheaves in Geometry and Logic*, Springer-Verlag, New York.

Mallios, A. (1998). *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry*, Vols. 1–2, Kluwer Academic Publishers, Dordrecht.

Mallios, A. (2002). Remarks on singularities (gr-qc/0202028).
Raptis, I. (2001). Presheaves, sheaves, and their topoi in quantum gravity and quantum logic (gr-qc/0110064).
Rawling, J. P. and Selesnick, S. A. (2000). Orthologic and quantum logic. Models and computational elements. *Journal of the Association for Computing Machinery* **47**, 721.
Takeuti, G. (1978). Two applications of logic to mathematics. *Mathematical Society of Japan* **13**, Kano Memorial Lectures 3.
Varadarajan, V. S. (1968). *Geometry of Quantum Mechanics, Vol. 1*, Van Nostrand, Princeton, NJ.